Velocity Scheduling Controller for a Nonholonomic Mobile Robot

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Abstract—An improvement over classical dynamic feedback linearization for the control of a nonholonomic mobile robot is proposed. The use of a state extension of higher dimension than in the case of dynamic feedback linearization helps reject constant disturbances on the rotational axis of the robot. The proposed dynamic extension acts as a velocity scheduler for the robot. It specifies at each time instant the ideal translational velocity that the robot should have. By having a two-dimensional state extension, both the magnitude and the orientation of the velocity vector can be generated, which accounts for improved robustness.

I. INTRODUCTION

This paper proposes a new control methodology for stabilizing a nonholonomic wheeled robot via trajectory tracking. Over the last 25 years, the nonholonomic mobile robot unicycle has been used as a case study for control (Canudas de Wit and Sordalen, 1992; Bloch, 2003; Rouchon et al., 1993; Jiang and Nijmeijer, 1997; Dixon et al., 2001) and stabilization (Kolmanovsky and McClamroch, 1995; Bloch, 1992) of nonholonomic dynamic systems. As is well-known now from Brockett’s theorem on necessary conditions for asymptotic stability (Brockett, 1983), the main difficulty in controlling nonholonomic systems is that there exists no continuous time-invariant state-feedback controllers that asymptotically stabilizes the system at any equilibrium point of interest.

We can classify the appropriate control methods into three categories: (i) discontinuous control laws, (ii) time-varying controllers, and (iii) dynamic extension and flatness-based controllers.

As far as the first category is concerned (Astolfi, 1996), discontinuous controllers can be extremely sensitive to actuator noise, and their robust design is a technically difficult issue that needs further investigations. Some preliminary results are available for a class of nonholonomic systems with uncertainties (Jiang, 2000; Bloch, 2003).

A major shortcoming of the smooth time-varying feedback controllers in the second category is that the closed-loop system is only asymptotically stable at the origin with no guarantee of exponential stability (Pomet, 1992; Samson, 1995). Nevertheless, it should be mentioned that exponential convergence can be achieved for a class of nonholonomic systems by means of Lipschitz continuous homogeneous time-varying feedback. However, as was shown in (Jiang, 2000), homogeneous feedback laws often do not guarantee stability in the presence of even small disturbances.

As for the third category of feedback laws, the techniques belong to two subclasses: a) dynamic feedback linearization (Oriolo et al., 2002), and b) feedforward control based on flatness (Fliess et al., 1995; Fliess et al., 1999; Rouchon et al., 1993). On the one hand, dynamic feedback linearization transforms the initial system into a linear and controllable form, for which classical linear controllers can be designed.

A drawback with this technique is that the controller cannot reject constant disturbances along the rotational axis of the robot (this will be illustrated later in this paper). On the other hand, although flatness is closely linked to dynamic feedback linearizability, it is applied differently. Flatness states a correspondence (Lie–Bäcklund diffeomorphism; for a formal definition see Fliess et al. (1999)) between the original state trajectories and the set of trajectories of particular outputs, termed the flat outputs. Then, by ensuring that the inputs follow trajectories compatible with those of the flat outputs, all states are under control, due to this correspondence. This property is used in the motion planning context, where an open-loop control is sought, so as to transfer the system from one initial state to another final one in finite time. In (Rouchon et al., 1993) and (Fliess et al., 1999), the initial and final configurations give rise to two specific initial and final values for the flat outputs and their derivatives. Then, by computing a suitable interpolating polynomial for these end conditions, an open-loop steering control is obtained as a function of the flat outputs and finite number of their time derivatives (i.e. without integrating differential equations). Nevertheless, if a perturbation occurs, a few options are available: i) use a local controller around this reference trajectory; ii) re-generate the reference trajectory if the perturbation is too serious; iii) use both a local controller and a re-generation mechanism in a separate way; iv) blend both the re-generating mechanism and the perturbation rejection controller. Clearly, the necessity of re-generating the interpolating polynomial so as to take into account the state drift inflicted by the perturbation is a critical issue, for it could very well deteriorate the overall behavior.
and induce instability in the system.

The method in this paper uses a two-dimensional state extension, contrary to classical feedback linearization that uses dynamic extension with a single state extension. Here, equivalence to the original system is enforced only asymptotically. Roughly stated, the concept is the following. The original system, together with the second-order dynamic extension, provides a reference heading angle that is tracked using a proportional controller. Upon tracking convergence, the extended 5th-order system (i.e. the original third-order system with the second-order extension) becomes equivalent to a 4th-order linear system that is easily stabilized. This should be contrasted with the classical dynamic extension approach, where the correspondence is enforced at every time instant.

The controller can also be interpreted in the light of the flatness property. A suitable trajectory re-generation mechanism of type iv) mentioned above is given by a filter, termed here velocity scheduling dynamic extension. Based on the feedback of the current robot position, the filter provides the corresponding value of the flat output, together with all the necessary time derivatives at every time instant. The input is then obtained much in the same way as for motion planning technique using flatness-based system inversion.

Despite its advantage over dynamic feedback linearization, this implicit re-generation mechanism necessitates a more involved stability analysis.

II. PRELIMINARIES

A. Nonholonomic Mobile Robot

Consider a mobile robot moving on a planar surface where \( x_1 \) is the horizontal coordinate and \( x_2 \) the vertical one. The angle that the robot makes with the horizontal axis is \( x_3 \). The kinematic equations of motion are given by:

\[
\begin{align*}
\dot{x}_1 &= v_1 \cos(x_3) \quad (1) \\
\dot{x}_2 &= v_1 \sin(x_3) \quad (2) \\
\dot{x}_3 &= v_2, \quad (3)
\end{align*}
\]

where \( v_1 \) and \( v_2 \) are the inputs. \( v_1 \) denotes the velocity in the direction defined by the heading angle and \( v_2 \) the angular velocity.

B. Dynamic Extension Controller

Dynamic feedback linearization, can be applied to the mobile robot described in the previous section. This has been undertaken in (Oriolo et al., 2002), where the system is transformed into two separate chains of integrators (of two integrators each) using dynamic extension. Each chain is then controlled by choosing the gains appropriately. This results in the controller

\[
\begin{align*}
\dot{\chi} &= -(k_1 \xi \cos(x_3) + k_2 x_1) \cos x_3 \\
&\quad - (k_3 \xi \sin x_3 + k_4 x_2) \sin x_3,
\end{align*}
\]

that sets the inputs to the robot:

\[
\begin{align*}
v_1 &= \xi \\
v_2 &= (k_1 \xi \cos x_3 + k_2 x_1) \sin x_3 \\
&\quad - (k_3 \xi \sin x_3 + k_4 x_2) \cos x_3.
\end{align*}
\]

C. Flatness Analysis

Choosing \( x_1 \) and \( x_2 \) as the flat outputs, i.e. \( y = (x_1, x_2)^T \), and using (1) and (2), \( x_3 \) can be expressed as:

\[
x_3 = \arctan(\dot{x}_1, \dot{x}_2) + \beta \pi \quad \beta = 0, 1 \quad (4)
\]

where \( \arctan(...) \) is the four quadrant arc-tangent function defined as a mapping from \( \mathbb{R} \times \mathbb{R} \) to \( [-\pi, \pi] \)

\[
\arctan(\dot{x}_1, \dot{x}_2) = \begin{cases} 
0 & \text{if } \dot{x}_1 = \dot{x}_2 = 0 \\
- i \ln(\frac{\dot{x}_1 + i \dot{x}_2}{\sqrt{\dot{x}_1^2 + \dot{x}_2^2}}) & \text{otherwise} \end{cases} \quad (5)
\]

Then, the inputs \( v_1 \) and \( v_2 \) become:

\[
\begin{align*}
v_1 &= (-1)\beta \sqrt{\dot{x}_1^2 + \dot{x}_2^2} \quad \beta = 0, 1 \quad (6) \\
v_2 &= -\dot{x}_2 \dot{x}_1 + \dot{x}_1 \dot{x}_2 \\
&\quad \dot{x}_1^2 + \dot{x}_2^2. \quad (7)
\end{align*}
\]

In view of (4), (6) and (7), the mobile robot model is clearly differentially flat according to the definition given in (Fliess et al., 1995).

D. S – \( \xi \) and \( S_{x_3} \) Systems

III. METHODOLOGY

A. Asymptotic Linearization

Inspired by the flatness analysis of the mobile robot recalled in Section II-C, the input \( v_1 \) can be expressed as \( \sqrt{\chi_1^2 + \chi_2^2} \) where \( \chi_1 \) and \( \chi_2 \) are ideal translational velocities (i.e. ideal \( \dot{x}_1 \) and \( \dot{x}_2 \)) that the robot should have. To achieve asymptotic convergence to the desired final position, it is necessary to both schedule adequately these ideal velocities and steer the robot angle \( x_3 \) accordingly. The scheduler updates the ideal velocities based on the current position of the robot (the scheduler will be given in (11) and (12)). The input \( v_2 \) is set as the output of a proportional regulator that makes \( x_3 \) converge to \( \dot{x}_3 = \arctan(\chi_1, \chi_2) \).

The interplay between the controller and the mobile robot ensures asymptotic convergence as stated in the following Proposition:

**Proposition 1:** Consider the system (1)-(3) with the following controller:

\[
\begin{align*}
\beta &= 0, 1 \quad (8) \\
v_1 &= (-1)^\beta \sqrt{\chi_1^2 + \chi_2^2} \quad (9) \\
v_2 &= \frac{\chi_2 \chi_1 - \chi_1 \chi_2}{\chi_1^2 + \chi_2^2} - k_\rho(\dot{x}_3 - \arctan(\chi_1, \chi_2) + \beta \pi),
\end{align*}
\]

\[
\begin{align*}
\dot{\chi}_1 &= k_1 x_1 + k_2 \chi_1 \\
\dot{\chi}_2 &= k_3 x_2 + k_4 \chi_2 \quad (10)
\end{align*}
\]
Pick negative parameters $k_i, 1 \leq i \leq 4$, such that 
\[ r_1 = \frac{k_1 + \sqrt{k_1^2 - 4k_2}}{2}, \quad r_2 = \frac{k_2 + \sqrt{k_2^2 - k_3k_4}}{2}, \quad r_3 = 1 + \frac{k_4 - \sqrt{k_4^2 + k_5^2}}{2} \] are negative real.

$x_3$ is given as $x_3 = x_3 + 2k\pi$ where $k$ is a discrete state determined as follows. Let $t_i$ denote any time instant for which the system states $\chi_1(t_i) < 0$ and $\chi_2(t_i) = 0$. Then, $k$ is updated according to the rules:
\[ k := k - 1 \quad \text{if} \quad \chi_2(t_i) < 0 \]
\[ k := k + 1 \quad \text{if} \quad \chi_2(t_i) > 0 \]
\[ k := k + \text{sgn}(k_3\sin x_3(t_i)) \quad \text{if} \quad x_2(t_i) = 0, \]
where we have used the fact that, by (12), $x_2(t_i) = 0$ if and only if $\chi_2(t_i) = 0$ when $\chi_2(t_i) = 0$. Under these conditions, $x_1$ and $x_2$ converge to zero exponentially. Moreover, if the poles are chosen such that $r_1 = r_2$, $r_3 = r_4$ and $r_1 > r_3$ with $k_p$ sufficiently large, the following two convergences are possible depending on the value of $\beta$. In case $\beta = 0$, then $x_3$ converges to
\[ \lim_{t \to \infty} x_3(t) = \begin{cases} -\pi & \text{if} \quad (-r_1x_1(0) + \dot{x}_1(0)) > 0 \\ 0 & \text{if} \quad (-r_1x_1(0) + \dot{x}_1(0)) < 0 \end{cases} \]
In case $\beta = 1$, then $x_3$ converges to
\[ \lim_{t \to \infty} x_3(t) = \begin{cases} -\pi & \text{if} \quad (-r_1x_1(0) + \dot{x}_1(0)) > 0 \\ 0 & \text{if} \quad (-r_1x_1(0) + \dot{x}_1(0)) < 0 \end{cases} \]

**Outline for the proof of Proposition 1:** The proof is split into three phases. Consider the dynamic system comprised of (1), (2), (3), (16), (17), with $v_1$ and $v_2$ as given in the proposition. This system, whose state space is $(x_1, \chi_1, x_2, \chi_2, x_3)^T$, is termed the $S_{x_3}$-system. The first phase of the proof shows that this system is diffeomorphic (i.e. there exists a change of coordinates given as a differentiable map with differentiable inverse) to another system named the $S_{\xi}$-system with state space $(x_1, \chi_1, x_2, \chi_2, \xi)^T$. However, the diffeomorphism holds for all $\chi_1$ and $\chi_2$ except at a specific set where a strong discontinuity occurs. The second phase then assesses properties of the $S_{\xi}$-system. It is shown that the states $x_1, x_2, \chi_1, \chi_2$ eventually decrease and converge exponentially to zero, despite a bounded initial transient (i.e. there is no finite escape time). The third phase then studies the properties of the $S_{x_3}$-system and, in particular, its behavior when crossing the transition set. Finally, asymptotic properties of $x_3$ are established. Without loss of generality, only the case with $\beta = 0$ is considered here. The three phases are detailed next.

**1) Phase I:** Combining the robot equations (1)-(3) and the dynamic extension with the inputs defined as (9)-(10), where $x_3$ is used instead of $\bar{x}_3$, leads to what will be called the $S_{x_3}$-system:
\[ \dot{x}_1 = \sqrt{x_1^2 + \chi_1^2}\cos x_3 \quad (13) \]
\[ \dot{\chi}_1 = k_1x_1 + k_2\chi_1 \quad (14) \]
\[ \dot{x}_2 = \sqrt{x_2^2 + \chi_2^2}\sin x_3 \quad (15) \]
\[ \dot{\chi}_2 = k_3x_2 + k_4\chi_2 \quad (16) \]
\[ \dot{x}_3 = \frac{\dot{x}_2\chi_1 - \dot{x}_2\chi_1}{\chi_1^2 + \chi_2^2} - k_p(x_3 - \arctan(\chi_1, \chi_2)). \quad (17) \]

Then, consider the change of coordinates between $\chi_1, \chi_2, x_3$ and $\chi_1, \chi_2, \xi$ ($x_1$ and $x_2$ do not influence) with $\xi$ given by:
\[ \xi = x_3 - \arctan(\chi_1, \chi_2) \quad (18) \]

Examining expression (5) shows that this change of coordinates is continuous except at the set $\{\chi_1 = 0, \chi_2 = 0\}$. Now, except on this set, and after defining the new state space $(x_1, \chi_1, x_2, \chi_2, \xi)^T$, the $S_{x_3}$-system is equivalent to This system is termed the $S_{\xi}$-system. Equivalence means here that solutions of the $S_{x_3}$-system are in diffeomorphic correspondence with those of the $S_{\xi}$-system. To check this, differentiate the change of coordinates (18) and use (1), (2), (11) and (12).

**2) Phase II:** Important properties of the $S_{\xi}$-system are given next. The first lemma deals with the absence of finite escape time. The second lemma shows that the $S_{\xi}$-system does indeed satisfy the hypothesis of this first lemma. Then, the states $x_1, x_2, \chi_2$ are shown to eventually converge exponentially to zero.

**Lemma 1:** Consider a cascade connected system of the form
\[ \dot{\chi} = F(\chi, \xi) \quad (19) \]
\[ \dot{\xi} = G(\xi) \quad (20) \]
with $\chi \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^m$. Assume that
\[ \dot{\chi} = F(\chi, 0) \]
is globally exponentially stable at $x = 0$ and
\[ \dot{\xi} = G(\xi) \]
is globally stable at $\xi = 0$. If the growth rate condition
\[ \|F(\chi, \xi) - F(\chi, 0)\| \leq c_1\|\xi\| + \psi(\|\xi\|) + c_2 \quad (22) \]
is satisfied for some constants $c_1, c_2 \geq 0$ and a nonnegative function $\psi$, then the cascade system does not exhibit any finite escape time.

**Proof:** Since system $\dot{\chi} = F(\chi, 0)$ is globally exponentially stable at $\chi = 0$, by the converse Lyapunov theorem (Khalil, 2002), there exists a positive definite and proper function $V$ such that
\[ \alpha_1\|\chi\|^2 \leq V(\chi) \leq \alpha_2\|\chi\|^2, \]
\[ \frac{\partial V}{\partial \chi} F(\chi, 0) \leq -\alpha_3\|\chi\|^2 \]
for some positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Then, by hypothesis and by completing the squares,
\[ \dot{V} = \frac{d}{dt} V(\chi(t)) = \frac{\partial V}{\partial \chi}(\chi(t))F(\chi, 0) + \frac{\partial V}{\partial \chi}(F(\chi, \xi) - F(\chi, 0)) \leq -\alpha_3\|\chi\|^2 + \alpha_4\|\chi\| (c_1\|\chi\| + \psi(\|\xi\|) + c_2) \leq c_4\|\chi\|^2 + \psi(\|\xi\|)^2 + c_5 \]
with appropriate nonnegative constants $c_4$ and $c_5$. Now, using
the hypothesis that the system $\xi = G(\xi)$ is globally stable, it
follows that $V(\chi(t))$ does not exhibit any finite escape time.
This, in turn, completes the proof of the lemma.

**Lemma 2:** The system (1)-(3) and the controller (9)-(12) can be put in the form (19), with
$\chi = (\chi_1, x_1, x_2, x_2)$, where $\chi_1$ and $\chi_2$ are given by (11), (12), and $\xi = x_3 - \arctan(\chi_1, \chi_2)$. Moreover, both subsystems $\hat{\chi} = F(\chi, 0)$ and $\hat{\xi} = G(\xi)$ are globally exponentially stable at the origin
and (22) is fulfilled with $c_1 \neq 0$, $\psi(||\xi||) = 0$ and $c_2 = 0$.

**Proof:** Using (1), (2), (11) and (12), $F(\chi, \xi)$ can be written:

$$F(\chi, \xi) = \left( \begin{array}{c} k_1x_1 + k_2\chi_1 \\ k_3x_2 + k_4\chi_2 \end{array} \right) \left( \begin{array}{c} \sqrt{\chi_1^2 + \chi_2^2 \cos(\xi + \arctan(\chi_1, \chi_2))} \\ \sqrt{\chi_1^2 + \chi_2^2 \sin(\xi + \arctan(\chi_1, \chi_2))} \end{array} \right)$$

Using (3) and (10), $G(\xi)$ is obtained as:

$$G(\xi) = -k_p\xi$$

(24)

from which follows that $\dot{\xi} = G(\xi)$ is exponentially stable.

When $\xi = 0$, and after some algebraic manipulations, $F(\chi, 0)$ becomes:

$$F(\chi, 0) = \left( \begin{array}{c} k_1x_1 + k_2\chi_1 \\ k_3x_2 + k_4\chi_2 \end{array} \right) \left( \begin{array}{c} \chi_1 \\ \chi_2 \end{array} \right)$$

(25)

As can be directly checked, $\hat{\chi} = F(\chi, 0)$ is globally exponentially stable at the origin.

Let

$$\triangle F(\chi, \xi) \triangleq F(\chi, \xi) - F(\chi, 0).$$

(26)

Now, (22) can be expressed using the canonical basis $e_i, i = 1, \ldots, 4$:

$$\|\triangle F(\chi, \xi)\| = \|\sum_{i=1}^{4} \triangle F(\chi, \xi)_i e_i\|$$

$$\leq \sum_{i=1}^{4} \|\triangle F(\chi, \xi)_i\| =$$

$$\leq \sqrt{\chi_1^2 + \chi_2^2 \cos(\xi + \arctan(\chi_1, \chi_2))} - \chi_1 +$$

$$\leq \sqrt{\chi_1^2 + \chi_2^2 \sin(\xi + \arctan(\chi_1, \chi_2))} - \chi_2 \|$$

$$\leq \|\chi_1\| + \|\chi_2\| \leq 3\|\chi\| \leq c_1\|\chi\|$$

(27)

Here $c_2 = 0$ and $\psi(||\xi||) = 0$.

**Lemma 3:** Let $P = P^T > 0$, $Q = Q^T > 0$ be such that,

$$P \left( \begin{array}{c} 0 & 1 \\ k_1 & k_2 \end{array} \right) + \left( \begin{array}{c} 0 & 1 \\ k_1 & k_2 \end{array} \right)^T P = -I_2,$$

$$Q \left( \begin{array}{c} 0 & 1 \\ k_3 & k_4 \end{array} \right) + \left( \begin{array}{c} 0 & 1 \\ k_3 & k_4 \end{array} \right)^T Q = -I_2.$$

Then,

$$V_0 = (x_1 \chi_1 + (x_2 \chi_2) Q (x_2 \chi_2)$$

converges to zero exponentially.

**Proof:** Notice first that

$$\dot{x}_1 = \chi_1 + \sqrt{\chi_1^2 + \chi_2^2} (\cos(\dot{x}_3 + \xi) - \cos(\dot{x}_3))$$

(28)

$$\dot{x}_1 = k_1x_1 + k_2\chi_1$$

(29)

$$\dot{x}_2 = \chi_2 + \sqrt{\chi_1^2 + \chi_2^2} (\sin(\dot{x}_3 + \xi) - \sin(\dot{x}_3))$$

(30)

$$\dot{x}_2 = k_3x_2 + k_4\chi_2$$

(31)

Then,

$$\dot{V}_0 = -(x_1^2 + \chi_1^2) - (x_2^2 + \chi_2^2)$$

$$+ 2(x_1 \chi_1) P\left( \begin{array}{c} 1 \\ 0 \end{array} \right) \sqrt{\chi_1^2 + \chi_2^2} \cdot (\cos(\dot{x}_3 + \xi) - \cos(\dot{x}_3))$$

$$+ 2(x_2 \chi_2) Q\left( \begin{array}{c} 1 \\ 0 \end{array} \right) \sqrt{\chi_1^2 + \chi_2^2} \cdot (\sin(\dot{x}_3 + \xi) - \sin(\dot{x}_3))$$

Since

$$\cos(\dot{x}_3 + \xi) - \cos(\dot{x}_3) = -\xi \int_0^1 \sin(\dot{x}_3 + \lambda \xi) d\lambda$$

$$\sin(\dot{x}_3 + \xi) - \sin(\dot{x}_3) = \xi \int_0^1 \cos(\dot{x}_3 + \lambda \xi) d\lambda,$$

it follows that

$$\dot{V}_0 = -(x_1^2 + \chi_1^2) - (x_2^2 + \chi_2^2)$$

$$- 2(x_1 \chi_1) P\left( \begin{array}{c} 1 \\ 0 \end{array} \right) \sqrt{\chi_1^2 + \chi_2^2} \cdot (x_1 \chi_1) \int_0^1 \sin(\dot{x}_3 + \lambda \xi) d\lambda$$

$$+ 2(x_2 \chi_2) Q\left( \begin{array}{c} 1 \\ 0 \end{array} \right) \sqrt{\chi_1^2 + \chi_2^2} \cdot \xi \int_0^1 \cos(\dot{x}_3 + \lambda \xi) d\lambda.$$

Now, the terms involving $P$ and $Q$ can be bounded as follows,

$$\|2(x_1 \chi_1) P\left( \begin{array}{c} 1 \\ 0 \end{array} \right) \sqrt{\chi_1^2 + \chi_2^2}\|$$

$$\leq (x_1^2 + 2\chi_1^2 + \chi_2^2) \|P\left( \begin{array}{c} 1 \\ 0 \end{array} \right)\| \|\xi\|,$$

and

$$\|2(x_2 \chi_2) Q\left( \begin{array}{c} 1 \\ 0 \end{array} \right) \sqrt{\chi_1^2 + \chi_2^2}\|$$

$$\leq (x_2^2 + 2\chi_2^2 + \chi_2^2) \|Q\left( \begin{array}{c} 1 \\ 0 \end{array} \right)\| \|\xi\|.$$

Obviously, the hypotheses of Lemma 1 are satisfied and, hence, there is no finite escape time. Therefore, picking $c > 0$ sufficiently small so that

$$1 - 2c \left( \|P\left( \begin{array}{c} 1 \\ 0 \end{array} \right)\| + \|Q\left( \begin{array}{c} 1 \\ 0 \end{array} \right)\| \right) > 0,$$

there will always exist a finite time $T$ for which

$$\|\xi(t)\| \leq c \quad \forall t \geq T.$$
Then, using the fact that
\[
\left| \int_0^1 \sin(\hat{x}_3 + \lambda \xi) d\lambda \right| \leq 1, \quad \left| \int_0^1 \cos(\hat{x}_3 + \lambda \xi) d\lambda \right| \leq 1,
\]
and noticing that both \(\frac{1}{2} x_3^2 + \frac{1}{2} x_2^2 + x_1^2 < x_3^2 + x_2^2 + x_1^2 + x_2^2\) and \(\frac{1}{2} x_3^2 + \frac{1}{2} x_2^2 + x_1^2 < x_3^2 + x_2^2 + x_1^2 + x_2^2\), there exists a positive constant \(c\) such that
\[
\dot{V}_0 \leq \left( -\frac{1}{2} + c \left( \| P \left( \frac{1}{0} \right) \| + \| Q \left( \frac{1}{0} \right) \| \right) \right) \\
\cdot (x_1^2 + x_2^2 + x_3^2 + x_2^2) \\
\leq -\dot{V}_0 \quad \forall t \geq T.
\]
(34)
Thus, one sees that \(V_0 \to 0\) exponentially.

3) Phase III: Both the \(S_{x_3}\)-system and the \(S_\xi\)-system have continuous-time solutions. This follows from the fact that solutions approaching the discontinuity from one side leave the discontinuity from another side, and therefore solutions can be defined as absolutely continuous functions satisfying the equation almost everywhere (Filippov, 1988).

Next, as long as the partial states \(\chi_1(t)\) and \(\chi_2(t)\) (corresponding to these solutions) do not enter the transition set \(\{\chi_1 < 0, \chi_2 = 0\}\) nor the exceptional set \(\{\chi_1 = 0, \chi_2 = 0\}\), the asymptotic property of the \(S_{x_3}\)-system can be directly deduced from the \(S_\xi\)-system. This follows from the diffeomorphic correspondence between solutions to these two systems starting from compatible initial conditions.

This is no longer the case when the solutions visit the transition set. The discontinuity appearing in the \(\text{arctan}\) function must be considered, by correcting the transition set. The discontinuity appearing in the square root cannot vanish. However, on the transition set, \(x_3(t) \to 0\) is impossible. This would mean, after examining (15), that \(\chi_2(t) = 0\) and \(x_2(t) = 0\) when the robot enters the exceptional manifold.

Therefore, even though \(\hat{x}_3\) might become arbitrarily large (the robot spins quickly on itself), this does not have dramatic consequences on the transversal position of the robot (both translational velocities vanish). Moreover, the system exits the exceptional manifold right away and restarts on a "regular" trajectory as will be shown in the following Lemma.

Lemma 6: Let \(\Phi(t)\) denote the solution to system (1-3) together with the controller (9-12), i.e. \(\Phi(t) = (\chi_1(t), \chi_2(t), x_1(t), x_2(t), x_3(t), x_3(t))\), where \(V = \{x \mid x_1 = 0, x_2 = 0, x_3 = 0\}\). If there exists a time instant \(t_1 > 0\) for which \(\Phi(t_1) \in W \setminus V\), then there exists an \(\epsilon > 0\) for which \(\Phi(t) \notin W \setminus V\) for all \(t_1 - \epsilon < t < t_1 + \epsilon, t \notin t_1\).

Proof: The proof consists in showing that the solution manifold is transversal to the set \(W \setminus V\). To see this, we will first show that \(T_{\Phi(t)}(x) \notin T_W\) for each \(x \in W \setminus V\). Then, since the vector field defining the dynamics is continuous, the above transversality is guaranteed, i.e. there exists a certain \(\epsilon > 0\) for which \(\Phi(t) \notin W\) for all \(t_1 - \epsilon < t < t_1 + \epsilon, t \notin t_1\).

The manifold \(W\) is defined by \(\chi_1 = 0\) and \(\chi_2 = 0\). Then, \(T_{\Phi(t)}(x)\) for \(x \in W\) is given by
\[
T_{\Phi(t)}(x) = (-k_1 x_1 - k_2 x_2 0 0 \beta(x))^T
\]
where \(x = (\chi_1, \chi_2, x_1, x_2, x_3)\) and \(\beta(x)\) is a corresponding scalar function of \(x\). Now \(x \in W\) means that \(\chi_1 = 0\) and \(\chi_2 = 0\). Hence,
\[
T_W = \text{span} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]
(36)
clearly exhibits the fact that, as long as both \( x_1 \) and \( x_2 \) do not cancel simultaneously, i.e. as long as \( x \in W \setminus V \), then \( T_{P_{(x)}}(x) \not\in T_W(x) \) and the conclusion follows.

Nevertheless, it remains to find out what is exactly happening to \( x_3 \) after crossing this exceptional manifold.

**Lemma 7:** Let \( t_i \) denote the time instant at which \( \chi_1(t_i) = 0 \) and \( \chi_2(t_i) = 0 \) with \( x_{3i} = x_3(t_i) \) not being both zero. Then, the following discontinuity in \( x_3 \) occurs:

\[
\lim_{t \to t_i^-} x_3(t_i + \epsilon) - \lim_{t \to t_i^+} x_3(t_i + \epsilon) = \pi \tag{37}
\]

Consider the \( S_2 \)-system when such a crossing occurs. Suppose it happens at time \( t_i \). This means \( x_1(t_i) = \bar{x}_1, x_2(t_i) = \bar{x}_2, \chi_1(t_i) = 0 \) and \( \chi_2(t_i) = 0 \), where at least \( \bar{x}_1 \) and \( \bar{x}_2 \) are non-zero. All states are well defined and continuous for the \( S_2 \)-system except possibly for their time derivative at \( t_i \) where discontinuities can occur. Since arctan is ill-defined at that instant time, the trajectory is split into two parts; one part for the time interval \( [t_i - \epsilon, t_i] \) and another for \( (t_i, t_i + \epsilon) \). For \( \epsilon \) small, only the first-order terms are considered, i.e. \( \xi(t) = (\xi(t_i) + \dot{\xi}(t_i)(t - t_i)) + O(\epsilon) \), \( x_1(t) = \bar{x}_1 \), \( x_2(t) = \bar{x}_2 \), \( \chi_1 = 0, \chi_2 = 0 \). Consider a small \( \delta > 0 \), for which

\[
\begin{align*}
\chi_1(t_i - \epsilon) &= \chi_1(t_i - \epsilon - \delta) + \xi_1(t_i - \epsilon) + O(\delta) \\
\chi_1(t_i + \epsilon + \delta) &= \chi_1(t_i + \epsilon) + \chi_1(t_i + \epsilon) + O(\delta).
\end{align*}
\]

Then, taking the limit \( \epsilon \to 0 \) so as to join both trajectory segments gives:

\[
\begin{align*}
\chi_1(t_i - \delta) &= -\chi_1(t_i) + O(\delta) = -k_1\bar{x}_1 + O(\delta) \\
\chi_1(t_i + \delta) &= \chi_1(t_i) + O(\delta) = k_1\bar{x}_1 + O(\delta).
\end{align*}
\]

A similar development can be undertaken for \( \chi_2 \). Thus, when approaching the manifold (prior to \( t_i \), \( x_3(t_i - \delta) = \xi(t_i - \delta) - k_1\bar{x}_1 - k_2\bar{x}_2 \), and when quitting the manifold (after \( t_i \), \( x_3(t_i + \delta) = \xi(t_i + \delta) + k_1\bar{x}_1 + k_2\bar{x}_2 \)). Therefore, by taking the limit \( \delta \to 0 \), the net difference is \( \pi \) and the result follows.

The next lemmas give the asymptotic value of \( x_3 \). By choosing conveniently the parameters of a reduced system given in Lemma 8, the final value of the coordinate associated to \( x_3 \) converges to zero without oscillating. Then, through setting \( k_p \) large enough, the asymptotic behavior of the \( S_2 \)-system can be made to match the one of the reduced system. Finally, by the diffeomorphic correspondance, the robot’s heading angle also converges to the desired final value.

**Lemma 8:** Given the system

\[
\frac{d}{dt} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} = \begin{pmatrix} k_1\bar{x}_1 + k_2\chi_1 \\ k_3\bar{x}_2 + k_4\chi_2 \end{pmatrix} \tag{38}
\]

with \( \bar{x}_3 = \arctan(\bar{x}_1, \bar{x}_2) \) and under the conditions that the poles of system (38) \( r_i, i = 1, \ldots, 4 \) are real negative and satisfy \( r_1 = r_2 = r_3 = r_4 \) and \( r_1 > r_3 \) then

\[
\lim_{t \to -\infty} \bar{x}_3(t) = \begin{cases} 
-\pi & \text{if } (-r_1\bar{x}_1(0) + \bar{x}_1(0)) > 0 \\
0 & \text{if } (-r_1\bar{x}_1(0) + \bar{x}_1(0)) < 0
\end{cases}
\]

**Proof:** Due to the linearity of System (38), when the poles are multiple, the solution \( \bar{x}_3(t) \) becomes:

\[
\begin{align*}
\bar{x}_1(t) &= e^{r_1t}(\bar{x}_1(0) + r_1t(-r_1\bar{x}_1(0) + \bar{x}_1(0))) \tag{39} \\
\bar{x}_2(t) &= e^{r_2t}(\bar{x}_2(0) + r_2t(-r_2\bar{x}_2(0) + \bar{x}_2(0))) \tag{40} \\
\bar{x}_3(t) &= e^{r_3t}(\bar{x}_3(0) + r_3t(-r_3\bar{x}_3(0) + \bar{x}_3(0))) \tag{41}
\end{align*}
\]

Now, the limit of \( \bar{x}_3 = \arctan(\bar{x}_1(t), \bar{x}_2(t)) \) for \( t \to \infty \) is considered. Substituting (39) and (41) in \( \arctan(\bar{x}_1(t), \bar{x}_2(t)) \) and after a few algebraic manipulations together with the fact that \( r_1 = r_2 = r_3 = r_4 \) and \( r_1 > r_3 \), the following result holds:

\[
\begin{align*}
\lim_{t \to \infty} \bar{x}_3(t) &= -t \ln \frac{-r_1\bar{x}_1(0) + \bar{x}_1(0)}{\sqrt{(-r_1\bar{x}_1(0) + \bar{x}_1(0))^2}} \\
\lim_{t \to \infty} \bar{x}_3(t) &= \lim_{t \to \infty} \arctan(\bar{x}_1(t), \bar{x}_2(t)) \tag{43} \\
\lim_{t \to \infty} \bar{x}_3(t) &= \begin{cases} 0 & \text{if } (-r_1\bar{x}_1(0) + \bar{x}_1(0)) > 0 \\
-\pi & \text{if } (-r_1\bar{x}_1(0) + \bar{x}_1(0)) < 0 \tag{44}
\end{cases}
\end{align*}
\]

**Lemma 9:** Let \( \chi(t) \) solution of system (45) and \( \tilde{\chi}(t) \) solution system of (44)

\[
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} k_1x_1 + k_2\chi_1 \\ k_3x_2 + k_4\chi_2 \\ \sqrt{x_1^2 + x_2^2} \cos(\xi + \arctan(\chi_1, \chi_2)) \end{pmatrix} \tag{45}
\]

If the initial conditions are chosen as \( \chi(0) = \tilde{\chi}(0) \) and \( \delta > 0 \) is an arbitrarily small positive number, then there exists a \( k_p \) sufficiently large such that

\[
\|\chi(t) - \tilde{\chi}(t)\| < \delta.
\]

**Proof:** Since

\[
\begin{align*}
\cos(\xi + \arctan(\chi_1, \chi_2)) &= \cos(\xi)\cos(\arctan(\chi_1, \chi_2)) \\
\sin(\xi + \arctan(\chi_1, \chi_2)) &= \sin(\xi)\sin(\arctan(\chi_1, \chi_2)) \tag{46}
\end{align*}
\]

the following identities are obtained

\[
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} k_1x_1 + k_2\chi_1 \\ \cos(\xi)\chi_1 - \sin(\xi)\chi_2 \\ k_3x_2 + k_4\chi_2 \end{pmatrix} \tag{47}
\]

Defining \( (1/k_p) = \epsilon \), (45) becomes:

\[
\epsilon \xi = -\xi.
\]

Eq. (47) represents a singularly perturbed system, so that results of this formalism can be applied to it. Indeed, all hypotheses of Theorem 9.1 of (Khalil, 2002) are verified, concluding that:

\[
\|\chi(t) - \tilde{\chi}(t)\| \leq c\epsilon|t| = \frac{c\epsilon}{k_p} = \delta.
\]
Lemma 10: By choosing $k_p$ sufficiently large, the system (1)-(3) with the controller of Proposition 1 ensures that

$$
\lim_{t \to \infty} x_3(t) = \begin{cases} 
-\pi & \text{if } -(r_1 x_1(0) + \dot{x}_1(0)) > 0 \\
0 & \text{if } -(r_1 x_1(0) + \dot{x}_1(0)) < 0
\end{cases}
$$

Proof: Let $\chi(t)$ be a solution of System (45) and $\hat{\chi}(t)$ be a solution to System (44). Consider the error between $\arctan(\hat{\chi}_1, \hat{\chi}_2)$ and $\arctan(\chi_1, \chi_2)$:

$$
\| \arctan(\hat{\chi}_1, \hat{\chi}_2) - \arctan(\chi_1, \chi_2) \| \leq \delta
$$

(48)

Thanks to Lemma 9,

$$
\chi_1 = \hat{\chi}_1 + w_1 \quad \| w_1 \| \leq \delta
$$

$$
\chi_2 = \hat{\chi}_2 + w_2 \quad \| w_2 \| \leq \delta
$$

(48) becomes

$$
\| \arctan(\hat{\chi}_1, \hat{\chi}_2) - \arctan(\hat{\chi}_1 + w_1, \hat{\chi}_2 + w_2) \| \leq \| \arctan(\hat{\chi}_1, \hat{\chi}_2) - \arctan(\hat{\chi}_1 + w_1, \hat{\chi}_2) \| + \| \arctan(\hat{\chi}_1 + w_1, \hat{\chi}_2) - \arctan(\hat{\chi}_1 + w_1, \hat{\chi}_2 + w_2) \| = \nu.
$$

Using the mean value theorem,

$$
\nu \leq \left\| \frac{\partial \arctan(\hat{\chi}_1, \hat{\chi}_2)}{\partial \hat{\chi}_1} \hat{\chi}_1 \right\| \| w_1 \| + \left\| \frac{\partial \arctan(\hat{\chi}_1, \hat{\chi}_2)}{\partial \hat{\chi}_2} \hat{\chi}_2 \right\| \| w_2 \| \leq 2\delta,
$$

with $\rho \in [0, 1]$. Thanks to (45) and the fact that $\xi = x_3 - \arctan(\chi_1, \chi_2)$, $x_3$ converges exponentially to $\arctan(\chi_1, \chi_2)$. Now, since $\| \arctan(\hat{\chi}_1, \hat{\chi}_2) - \arctan(\chi_1, \chi_2) \|$ is smaller than $2\delta$, $x_3$ converges to $\arctan(\hat{\chi}_1, \hat{\chi}_2)$ with an error smaller than $2\delta$. By choosing a sufficiently large $k_p$, $\delta$ can be reduced. Without loss of generality, from now on, only the case $-r_1 x_1(0) + \dot{x}_1(0) > 0$ is considered. Suppose $\lim_{t \to \infty} x_3(t) = c \neq 0$ with $c < 2\delta$. The diffeomorphic correspondence implies that $\xi$ converges to $c - \arctan(\chi_1, \chi_2)$. But then, since $\lim_{t \to \infty} x_i = \lim_{t \to \infty} \chi_i$, $i = 1, 2$, this would mean $\lim_{t \to \infty} \xi \neq 0$, contradicting the differential equation $\hat{\xi} = -k_p \xi$. Therefore, $c = 0$.

The proof of Proposition 1 can now be completed by putting together the various lemmas and propositions. Lemma 1 guarantees that there is no finite escape time in the solutions of the $S_\xi$-system. This is also the case for the $S_{x_3}$-system, as long as the solutions to each of these systems are in diffeomorphic correspondence. This is true outside the transition and exceptional sets. Under such circumstances, Lemmas 1, 2 and 3 together with the diffeomorphic correspondence guarantee that all states $x_1$, $x_2$, $\xi_1$, $\xi_2$ and $x_3$ decrease. However, when crossing the transition set, the implicit resetting mechanism (see Lemma 4 and Remark 1, which is obtained indirectly using the index $k$ in Proposition 1), shows that solutions can be defined so as to maintain the diffeomorphic correspondence. Nevertheless, the problem of visiting the exceptional set must still be clarified. Lemma 5 shows that, when crossing this set, only $x_3$ is affected and at most with a finite jump of $\pm\pi$, and therefore this does not affect the convergence process (only a finite number of crossings of the singularity has been witnessed; this observation can be made rigorous by considering the amount of time needed for the robot to be in a position to cross the singularity again). Finally, Lemma 10 guarantees that the angle converges adequately.

IV. SIMULATION

The stabilizing behavior of the velocity scheduling controller is now tested and compared with dynamic feedback linearization. The goal is to bring the mobile robot to the origin, i.e. $(x_1 = 0, x_2 = 0)$. The initial conditions and parameters of the controllers are given in Table I.

The first simulation is carried out in the absence of perturbation acting on the rotational axis of the robot. This is illustrated in Figure 1. The resulting trajectories are very similar and all states converge adequately in both cases.

For the second experiment, the robot model is changed to

$$
\begin{align*}
\dot{x}_1 &= v_1 \cos x_3 \\
\dot{x}_2 &= v_1 \sin x_3 \\
\dot{x}_3 &= v_2 + \delta,
\end{align*}
$$

where $\delta$ is a constant but unknown perturbation parameter. For the simulation undertaken, the value $\delta = 1$ has been chosen. The results are given in Figure 2. Although all translational positions go to zero in both cases, the classical dynamic feedback linearization controller cannot handle the perturbation along the rotational axis, and the robot keeps turning on itself. On the other hand, the velocity scheduling controller maintains robustness of convergence.

![Figure 1](image-url)

**Fig. 1.** When no perturbation acts on the rotational axis of the robot, classical dynamic feedback linearization (plain line) and velocity scheduling controller (dashed line) give roughly the same behavior (compatible initial conditions were provided).

The improvement can be explained as follows: Setting $v_2$ according to Proposition 1, leads to the $\xi$ equation of the $S_\xi$-system becoming $\dot{\xi} = -k_p \xi + \delta$. Therefore, $\xi$ converges to a constant which can be made as small as desired by increasing the gain $k_p$. Therefore, it leads to arbitrary ultimate boundedness in $\xi$. By means of total stability arguments, it is not hard to see that the lemmas can be adapted to conclude that $x_1, x_2$ are also practically stable.
A flatness-based dynamic extension type controller has been proposed, labeled velocity scheduling controller. Contrary to dynamic feedback linearization, where a single dimensional state extension is used, a two state dynamic extension is proposed. The main advantage lies in rejecting constant disturbances along the $x_3$ axis, as was illustrated in simulation.

The proof of convergence shows several difficulties. The first one is due to only asymptotic equivalence to a linear system, which implies that the convergence analysis is more difficult than for classical dynamic feedback linearization. It is necessary to guarantee that the mismatch between the nonlinear system and the asymptotic linear one can be bounded, before asymptotic convergence can be proved.

The second difficulty is the singularity crossing. It has been shown that, although the velocity along the rotational axis can become very large, the robot angle does not grow unbounded. The shift in the angle is shown to be exactly $\pm \pi$ when crossing the singularity.

Future work will address the application of the proposed methodology to a wider class of kinematic nonholonomic systems such as car with trailers, and flat underactuated mechanical systems such as cranes.

**V. Conclusion**

A flatness-based dynamic extension type controller has been proposed, labeled velocity scheduling controller. Contrary to dynamic feedback linearization, where a single dimensional state extension is used, a two state dynamic extension is proposed. The main advantage lies in rejecting constant disturbances along the $x_3$ axis, as was illustrated in simulation.

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